In Variance

Definition:

This property states that if θi an unbiased estimator of θ.then the same method leads to g(θ^*) is the estimator of g(θ).

Let a family of function $[F[\theta, \theta \in \theta]]$ is invariance (the distribution of $g(x)$ is same as the distribution of x), then an estimator is said to be invariance if

$T(g(x_1), g(x_2)...g(x_n)) = T(x_1, x_2, x_3,....x_n)$

Then there are two types of in-variance

i) Location invariance

ii)Parameter invariance

Location invariance

An estimator is defined to be location invariance if and only if
$T(x_1 + c, x_2 + c, x_3 + c,....x_n + c) = T(x_1, x_2, x_3,....x_n)$ for all x $_i$ + and c.

Procedure

If $T(x_1, x_2, x_3,....x_n)$= sample mean= $\frac{\sum x}{n}$

$$T(x_1, x_2, x_3,....x_n) = \frac{(x_1 , x_2 , x_3 , x_4 ..... x_n)}{n}$$

Then

$$T(x_1 + c, x_2 + c, x_3 + c,....x_n + c) = \frac{(x_1 + c , x_2 + c , x_3 + c , x_4 + c ..... x_n + C)}{n}$$

$$T(x_1 + c, x_2 + c, x_3 + c,....x_n + c) = \frac{(x_1 , x_2 , x_3 , x_4 ..... x_n)}{n} + \frac{nc}{n}$$

$$T(x_1 + c, x_2 + c, x_3 + c,....x_n + c) = \frac{\sum x}{n} + c$$

$$T(x_1 + c, x_2 + c, x_3 + c,....x_n + c) = \bar{x} + c$$

$$T(x_1 + c, x_2 + c, x_3 + c,....x_n + c) = T(x_1, x_2, x_3....x_n) + c$$

Then sample mean is called location invariance.

Scale in-variance

An estimator $T(x_1, x_2, x_3....x_n)$ is defined to be scale invariance if and only if $T(x_1c, x_2c, x_3c....x_nc)$ = $cT(x_1, x_2, x_3....x_n)$for all x $_i$ an c

Then $T(x_1, x_2, x_3....x_n)$ is called scale invariance.

Pitmen estimator for location

Let $T(x_1, x_2, x_3....x_n)$be a random sample from density f (x,θ) then an estimator

$$T(x_1, x_2, x_3....x_n) = \frac{\int_{-\infty}^{\infty} \theta \prod_1^n f(x,\theta) d\theta}{\int_{-\infty}^{\infty} \prod_1^n f(x,\theta) d\theta}$$

Is called pitman estimator for location and θ is called location in parameter.

Pitmen estimator for scale:

Let $(x_1, x_2, x_3....x_n)$ be a random sample from density f (x,θ) means θ>0 is a scale parameter assume that f (x,θ)=0 for x ≤ 0 i.e then random variable for assume only positive values within the class of invariant estimators

Then the estimator

$$T(x_1, x_2, x_3....x_n) = \frac{\int_{-\infty}^{\infty} \frac{1}{\theta^2} \prod_{1}^{n} f(x,\theta) d\theta}{\int_{-\infty}^{\infty} \frac{1}{\theta^2} \prod_{1}^{n} f(x,\theta) d\theta}$$

It is called pitman estimator for scale.

Location parameter

Let θ is called the location parameter for the density f (x,θ) of a random variable c if and only if the density "f $(x-\theta)$" does not depends on parameter θ then density function f (x,θ) can be written in the form of function $(x-\theta)$

If $\mathbf{f}(x,\theta) = \mathbf{f}(x-\theta)$

Scale parameter

Let θ is called the location parameter for the density f (x,θ) of a random variable c if and only if the distribution of $\left(\frac{x}{\theta}\right)$ is independent parameter of θ.

Q.No.1

Check whether or not the following estimator are location in variant

i) $\bar{x}$

$$= \frac{\sum x}{n}$$

$$= \frac{(x_1 + x_2 + x_3 + x_4 + \ldots + x_n)}{n}$$

Then by definition of location:

$$T(x_1 + c, x_2 + c, x_3 + c....x_n + c) = \frac{\sum(x + C)}{n}$$

$$T(x_1 + c, x_2 + c, x_3 + c....x_n + c) = \frac{\sum x + \sum c}{n}$$

$$T(x_1 + c, x_2 + c, x_3 + c....x_n + c) = \frac{\sum x}{n} + \frac{nc}{n}$$

$$T(x_1 + c, x_2 + c, x_3 + c....x_n + c) = T(x_1, x_2, x_3...x_n) + c$$

$$T(x_1 + c, x_2 + c, x_3 + c....x_n + c) = \bar{x} + c$$

Hence $\bar{x} = \frac{\sum x}{n}$ is an location in-variant .

ii) $\frac{y_{(1)} + y_{(n)}}{2}$

Let $T(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) = \dfrac{y_{(1)} + y_{(n)}}{2}$

$$T(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) = \frac{1}{2}[\min(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) + \max(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n})]$$

Then by definition of location:

$$T(\mathrm{y}_1 + \mathrm{c}, \mathrm{y}_2 + \mathrm{c}, \mathrm{y}_3 + \mathrm{c}, \mathrm{y}_4 + \mathrm{c} \ldots\ldots \mathrm{y}_\mathrm{n} + \mathrm{c}) = \frac{1}{2}[\min(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) + c + \max(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) + c]$$

$$T(\mathrm{y}_1 + \mathrm{c}, \mathrm{y}_2 + \mathrm{c}, \mathrm{y}_3 + \mathrm{c}, \mathrm{y}_4 + \mathrm{c} \ldots\ldots \mathrm{y}_\mathrm{n} + \mathrm{c}) = \frac{1}{2}[\min(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) + \max(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) + 2c]$$

$$T(\mathrm{y}_1 + \mathrm{c}, \mathrm{y}_2 + \mathrm{c}, \mathrm{y}_3 + \mathrm{c}, \mathrm{y}_4 + \mathrm{c} \ldots\ldots \mathrm{y}_\mathrm{n} + \mathrm{c}) = \frac{y_{(1)} + y_{(n)}}{2} + \frac{2c}{2}$$

$$T(\mathrm{y}_1 + \mathrm{c}, \mathrm{y}_2 + \mathrm{c}, \mathrm{y}_3 + \mathrm{c}, \mathrm{y}_4 + \mathrm{c} \ldots\ldots \mathrm{y}_\mathrm{n} + \mathrm{c}) = \frac{1}{2}\left[y_{(1)} + y_{(n)} + 2c\right] =$$

$$T(\mathrm{y}_1 + \mathrm{c}, \mathrm{y}_2 + \mathrm{c}, \mathrm{y}_3 + \mathrm{c}, \mathrm{y}_4 + \mathrm{c} \ldots\ldots \mathrm{y}_\mathrm{n} + \mathrm{c}) = \frac{1}{2}\left[y_{(1)} + y_{(n)}\right] + c$$

$$T(\mathrm{y}_1 + \mathrm{c}, \mathrm{y}_2 + \mathrm{c}, \mathrm{y}_3 + \mathrm{c}, \mathrm{y}_4 + \mathrm{c} \ldots\ldots \mathrm{y}_\mathrm{n} + \mathrm{c}) = T(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4 \ldots\ldots \mathrm{y}_\mathrm{n}) + c$$

Hence $\dfrac{y_{(1)} + y_{(n)}}{2}$ is location invariant.

iii) $\sqrt{\chi^2 - 1}$

Let $T(x_1, x_2, x_3 \ldots x_n) = \sqrt{\chi^2 - 1}$

Let by definition:

$$T(x_1 + c, x_2 + c, x_3 + c \ldots . x_n + c) = \sqrt{(\chi^2 + c)^2 - 1}$$

$$T(x_1 + c, x_2 + c, x_3 + c \ldots . x_n + c) = \sqrt{(\chi^2 + c^2 - 2cx) - 1}$$

$$T(x_1 + c, x_2 + c, x_3 + c \ldots . x_n + c) \neq T(x_1, x_2, x_3 \ldots . x_n) + c$$

Hence it is not location in variant

iv) $s^2 = \dfrac{\sum (x - \bar{x})^2}{n}$

Let

$$T(x_1, x_2, x_3 \ldots x_n) = \frac{\sum (x - \bar{x})^2}{n}$$

$$T(x_1, x_2, x_3 \ldots . x_n) = \frac{\sum \left(x - \frac{\sum x}{n} \right)^2}{n}$$

Then by definition

$$T(x_1 + c, x_2 + c, x_3 + c \ldots . x_n + c) = \frac{1}{n} \sum \left[(x + c) - \frac{\sum (x + c)}{n} \right]^2$$

$$T(x_1 + c, x_2 + c, x_3 + c \ldots . x_n + c) = \frac{1}{n} \sum \left[(x + c) - \frac{\sum (x)}{n} - \frac{\sum c}{n} \right]^2$$

$$T(x_1+c, x_2+c, x_3+c....x_n+c) = \frac{1}{n}\sum\left[x+c-\bar{x}-c\right]^2$$

$$T(x_1+c, x_2+c, x_3+c....x_n+c) = \frac{\sum(x-\bar{x})^2}{n}$$

$$T(x_1+c, x_2+c, x_3+c....x_n+c) \neq \mathrm{T}(x_1, x_2, x_3, x_4 ..... x_n) + c$$

Hence $s^2 = \frac{\sum(x-\bar{x})^2}{n}$ is not location invariance.

v) $y_{(n)} - y_{(1)}$

Let $T(y_1, y_2, y_3...y_n) = [\max(y_n) - \min(y_1)]$

$$T(y_1, y_2, y_3...y_n) = [\max(y_1, y_2, y_3, y_4 ..... y_n) - \min(y_1, y_2, y_3, y_4 ..... y_n)]$$

Then by definition

$$T(y_1+c, y_2+c, y_3+c, -.. y_n+c) = \max(y_1+c, y_2+c, y_3+c, -.. y_n+c) - \min(y_1+c, y_2+c, y_3+c, -.. y_n+c)$$
$$T(y_1+c, y_2+c, y_3+c, -.. y_n+c) = \max(y_1, y_2\ y_3, .. y_n) + c - \min(y_1, y_2, y_3, -.. y_n) - c$$

$$T(y_1+c, y_2+c, y_3+c, -.. y_n+c) = \max(y_1, y_2\ y_3, .. y_n) - \min(y_1, y_2, y_3, -.. y_n)$$

$$T(y_1+c, y_2+c, y_3+c, -.. y_n+c) \neq T(y_1, y_2\ y_3, .. y_n) + c$$

Hence $y_{(n)} - y_{(1)}$ is not location invariant .s

<u>Question 2:</u>

Check whether or not the following estimator are scale invariant.

i) $\bar{x}$

Let

$$\mathrm{T}(x_1, x_2, x_3, x_4 ..... x_n) = \frac{(x_1 + x_2 + x_3 + x_4 + ... + x_n)}{n} = \frac{\sum x}{n}$$

Then by definition of scale invariant

$$T(cx_1, cx_2, cx_3...cx_n) = \frac{\sum(cx)}{n}$$

$$T(cx_1, cx_2, cx_3...cx_n) = c.\frac{\sum x}{n}$$

$$T(cx_1, cx_2, cx_3...cx_n) = c\bar{x}$$

$$T(cx_1, cx_2, cx_3...cx_n) = cT(x_1, x_2, x_3...x_n)$$

Hence $\bar{x}$ is an scale invariant.

ii) $\bar{X}^2$

Let $T(x_1, x_2, x_3....x_n) = \overline{x^2}$

$$T(x_1, x_2, x_3....x_n) = \left(\frac{\sum x}{n}\right)^2$$

Then by definition:

$$T(x_1c, x_2c, x_3c \dots x_nc) = \left(\frac{c\bar{x}}{n}\right)^2$$

$$T(x_1c, x_2c, x_3c \dots x_nc) = c^2\left(\frac{\sum x}{n}\right)^2$$

$$T(x_1c, x_2c, x_3c \dots x_nc) = c^2\,\overline{x^2}$$

$$T(x_1c, x_2c, x_3c \dots x_nc) = c^2T(x_1, x_2, x_3 \dots x_n)$$

$$T(x_1c, x_2c, x_3c \dots x_nc) \neq cT((x_1, x_2, x_3 \dots x_n))$$

Hence $\bar{x}^2$ is not a scale invariant.

iii) s^2

Let

$$T(x_1, x_2, x_3 \dots x_n) = \frac{\sum(x-\bar{x})^2}{n}$$

$$T(x_1, x_2, x_3 \dots x_n) = \frac{1}{n}\sum\left(x - \frac{\sum x}{n}\right)^2$$

Then by definition

$$T(x_1c, x_2c, x_3c \dots x_nc) = \frac{1}{n}\sum\left[(xc) - \frac{\sum(xc)}{n}\right]^2$$

$$T(x_1c, x_2c, x_3c \dots x_nc) = c^2\,\frac{1}{n}\sum\left[(x) - \frac{\sum(x)}{n}\right]^2$$

$$T(x_1c, x_2c, x_3c \dots x_nc) = c^2T(x_1, x_2, x_{3.} \dots x_n)$$

$$T(x_1c, x_2c, x_3c \dots x_nc) \neq \mathrm{cT}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4 \dots \mathrm{x}_n)$$

Hence s^2 is not scale invariant.

iv) $\sqrt{s^2}$

$$T(x_1, x_2, x_3 \dots x_n) = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$T(x_1, x_2, x_3 \dots x_n) = \sqrt{\frac{\sum\left(x - \frac{\sum x}{n}\right)^2}{n}}$$

Then by definition

$$T(x_1c, x_2c, x_3c \dots x_nc) = \sqrt{\frac{1}{n}\sum\left[(xc) - \frac{\sum(xc)}{n}\right]^2}$$

$$T(x_1c, x_2c, x_3c \dots x_nc) = \sqrt{c^2\,\frac{1}{n}\sum\left[(x) - \frac{\sum(x)}{n}\right]^2}$$

$$T(x_1c, x_2c, x_3c....x_nc) = c\sqrt{\frac{1}{n}\sum\left[(x) - \frac{\sum(x)}{n}\right]^2}$$

$$T(x_1c, x_2c, x_3c....x_nc) = cT(x_1, x_2, x_{3.}....x_n)$$

Hence s is scale invariant.

v) $\frac{y_1 + y_n}{2}$

$$\text{Let}\left(\text{T}(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n\right) = \frac{y_{(1)} + y_{(n)}}{2}$$

$$= \frac{1}{2}[\min(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n) + \max(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n)]$$

Then by definition:

$$= \frac{1}{2}[\min(\text{y}_1\text{c}, \text{y}_2\text{c}, \text{y}_3\text{c}, \ldots \text{y}_n) + \max(\text{y}_1\text{c}, \text{y}_2\text{c}, \text{y}_3\text{c}, \ldots \text{y}_n)]$$

$$= \frac{1}{2}[c.\min(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n) + c.\max(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n)]$$

$$= \frac{1}{2}[y_{(1)} + y_{(n)}] = c\left(\frac{y_{(1)} + y_{(n)}}{2}\right)$$

$$= cT(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n)$$

Hence $\frac{y_{(1)} + y_{(n)}}{2}$ is scale invariant.

v) $y_{(n)} - y_{(1)}$

let $T(\text{y}_1, \text{y}_2, \text{y}_3 \ldots\ldots \text{y}_n) = [\max(\text{y}_n) - \min(\text{y}_1)]$

$$T(\text{y}_1, \text{y}_2, \text{y}_3 \ldots\ldots \text{y}_n) = [\max(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n) - \min(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n)]$$

Then by definition

$$T(\text{y}_1\text{c}, \text{y}_2\text{c}, \text{y}_3\text{c}, \text{y}_4\text{c} \ldots\ldots \text{y}_n c) = \max(\text{y}_1\text{c}, \text{y}_2\text{c}, \text{y}_3\text{c}, \text{y}_4\text{c} \ldots\ldots \text{y}_n c) - \min(\text{y}_1\text{c}, \text{y}_2\text{c} \ldots \text{y}_n\text{c})$$

$$T(\text{y}_1, \text{y}_2, \text{y}_3 \ldots\ldots \text{y}_n) = C\max(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n) - C\min(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n)$$

$$T(\text{y}_1, \text{y}_2, \text{y}_3 \ldots\ldots \text{y}_n) = C[\max(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n) - \min(\text{y}_1, \text{y}_2, \text{y}_3, \text{y}_4 \ldots\ldots \text{y}_n)]$$

$$T(\text{y}_1, \text{y}_2, \text{y}_3 \ldots\ldots \text{y}_n) = c(y_n - y_1)$$

Hence $y_{(n)} - y_{(1)}$ is scaleinvariant .

Question 3

If x~ N$(\theta,1)$then show that "θ" of $f(x)$is location parameter.

As x~ N$(\theta,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}$$

Differentiate function is

$$\text{df}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}$$

put $y = x - \theta$

dy=dx

df $y = (x-\theta) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y)^2}$ dy

$$f(x-\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y)^2}$$

Hence it is independent from parameter θ therefore θ is a location parameter.

Question no 4:

If α of Cauchy distribution having pdf $f(x) = \left[\dfrac{1}{\pi\left(1+(x-\alpha)^2\right)}\right]$ is location parameter or not?

As $f(x) = \left[\dfrac{1}{\pi\left(1+(x-\alpha)^2\right)}\right]$

Differential function is:

$$df(x) = \left[\frac{1}{\pi\left(1+(x-\alpha)^2\right)}\right] dx$$

Put $z = x - \alpha$

$x = \alpha + z$

$dx = dz$

$$df(z = x-\alpha) = \frac{1}{\pi(1+z)^2} dz$$

Hence it is independent from parameter α therefor $f(x)$ is a location parameter.

Question no5:

If $x \sim N(0, \theta^2)$ then show that θ is a scale parameter?

As $x \sim N(0, \theta^2)$

$$f(x) = \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{\theta^2}}$$

Differential function is

$$df(x) = \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{\theta^2}} dx$$

Put $y = \dfrac{x}{\theta}$ $\qquad\qquad$ $x = \theta y$

$\theta dy = dx$

$$df\left(y = \frac{x}{\theta}\right) = \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{y^2}{2}} \theta dy$$

$$df\left(y = \frac{x}{\theta}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

Hence it is independent from parameter θ therefore θ is scale parameter.

Question no 6

If $x \sim \exp$ then show that θ is a scale parameter.

As $f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$

Differential function is:

$$df(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$$

Put $\frac{x}{\theta} = t$ $\quad x = \theta t \quad dx = \theta dt$

$$= df\left(t = \frac{x}{\theta}\right) = \frac{1}{\theta} e^{-t} \theta dt$$

$$= e^{-t} dt$$

$$f\left(t = \frac{x}{\theta}\right) = e^{-t}$$

Hence it is independent from θ therefore θ is a scale parameter.

Question no 7

Obtain the scale parameter of the following function:

i) $f(x) = \frac{1}{\theta}$

As $f(x) = \frac{1}{\theta}$ $\quad -\infty \leq x \leq \infty$

$$df(x) = \frac{1}{\theta} dx$$

Put $\frac{x}{\theta} = y$ $\quad x = \theta y \quad dx = \theta dy$

$$df(x) = \frac{1}{\theta} \theta dy$$

$$df(x) = dy$$

Hence it is scale parameter.

ii) $f(x) = \frac{1}{2\theta - \theta}$ $\quad \theta \leq x \leq 2\theta$

As

$$f(x) = \frac{1}{2\theta - \theta}$$

$$= \frac{1}{\theta}$$

$df(x) = \frac{1}{\theta} dx$

Put $y = \frac{x}{\theta}$ $\quad x = y\theta \quad dx = \theta dy$

$df(x) = \frac{1}{\theta} \theta dy$

$df(x) = dy$

It is scale parameter.

iii) $f(x) = \frac{1}{\left(\frac{1}{\theta}\right)^{\mu}} x^{\mu-1} . e^{-\frac{x}{\theta}}$ $\quad 0 \le x \le \infty$

as $f(x) = \frac{1}{\left(\frac{1}{\theta}\right)^{\mu}} x^{\mu-1} . e^{-\frac{x}{\theta}}$

$df(x) = \frac{1}{\left(\frac{1}{\theta}\right)^{\mu}} x^{\mu-1} . e^{-\frac{x}{\theta}} dx$

Put

$y = \frac{x}{\theta}$ $\quad x = y\theta \quad dx = \theta dy$

$df\left(y = \frac{x}{\theta}\right) = \theta^{\mu} x^{\mu-1} . e^{-y} \ \theta dy$

$\theta^{\mu} (\theta y)^{\mu-1} . e^{-y} \theta dy$

$\theta^{\mu} (\theta)^{\mu-1} y^{\mu-1} . e^{-y} \theta dy$

$\theta^{\mu} (\theta)^{\mu-1+1} y^{\mu-1} . e^{-y} dy$

$\theta^{2\mu} \ y^{\mu-1} . e^{-y} dy$

Hence it is not a scale parameter .

Question no 8:

Let(x_1, x_2, x_3, x_4 ….. x_n)be a random sample from density $N(\theta,1)$. then show that $\bar{x}$ is pitman estimator for θ.(pitman estimator for location)

As we know that pitman estimator for location :

$$T(x_1, x_2, x_3, \ldots x_n) = \frac{\int_{-\infty}^{\infty} \theta \prod_{i=1}^{n} f(x,\theta) d\theta}{\int_{-\infty}^{\infty} \prod_{i=1}^{n} f(x,\theta) d\theta}$$

As

$x \sim N(0,1)$

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}$$

Then taking liklihoodfunction :

$$\prod_{i=1}^{n} f(x,\theta)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}$$

$$=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x-\theta\ )^2}$$

$$=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum\left[x^2+\theta^2-2\theta x\right]}$$

$$=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum\left[x^2+\theta^2-2\theta x\right]}$$

$$=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\left[\sum x^2+n\theta-2\theta\sum x\right]}$$

$$=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\left[\sum x^2+n\theta^2-2n\theta\bar{x}\right]} \qquad \therefore \bar{x}=\frac{\sum x}{n}$$

$$=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{\sum x^2}{2}} e^{-\frac{n\theta^2}{2}} e^{n\theta\bar{x}}$$

now

$$T(x_{1,}x_2,x_3,\ldots x_n)=\frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{-\infty}^{\infty}\theta e^{-\frac{\sum x^2}{2}} e^{-\frac{n\theta^2}{2}} e^{n\theta\bar{x}}d\theta}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{-\infty}^{\infty} e^{-\frac{\sum x^2}{2}} e^{-\frac{n\theta^2}{2}} e^{n\theta\bar{x}}d\theta}$$

$$=\frac{\int_{-\infty}^{\infty}\theta e^{-\frac{n}{2}\left(\theta^2-2\theta\bar{x}\right)}d\theta}{\int_{-\infty}^{\infty} e^{-\frac{n}{2}\left(\theta^2-2\theta\bar{x}\right)}d\theta}$$

Add and subtract $\bar{x}^2$

$$=\frac{\int_{-\infty}^{\infty}\theta e^{-\frac{n}{2}\left(\theta^2-2\theta\bar{x}+\bar{x}^2-\bar{x}^2\right)}}{\int_{-\infty}^{\infty} e^{-\frac{n}{2}\left(\theta^2-2\theta\bar{x}+\bar{x}^2-\bar{x}^2\right)}}d\theta$$

$$=\frac{e^{\frac{n\bar{x}^2}{2}\int_{-\infty}^{\infty}\theta e^{-\frac{n}{2}(\theta-\bar{x})^2}}d\theta}{e^{\frac{n\bar{x}^2}{2}\int_{-\infty}^{\infty} e^{-\frac{n}{2}(\theta-\bar{x})^2}}d\theta}$$

$$= \frac{\frac{1}{\sqrt{\frac{1}{n}}\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta e^{-\frac{1}{2\left(\frac{1}{n}\right)}(\theta-\bar{x})^2} d\theta}{\frac{1}{\sqrt{\frac{1}{n}}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\left(\frac{1}{n}\right)}(\theta-\bar{x})^2} d\theta}$$

Hence total area under the curve is 1.

Q.2: Let "X" be a random sample of size "n" from Possion distribution. Find the pitman estimator for scale parameter.

Solution:

As

$$T(X_1, X_2, X_3, \ldots, X_n) = \frac{\int_{-\infty}^{\infty} \frac{1}{\theta^2} \underset{i=1}{\overset{n}{\pi}} f(X_i, \theta) d\theta}{\int_{-\infty}^{\infty} \frac{1}{\theta^3} \underset{i=1}{\overset{n}{\pi}} f(X_i, \theta) d\theta}$$

$$f(x) = \frac{e^{-\theta}\theta^x}{x!} \qquad\qquad 0 < X < \infty$$

$$L(\underline{X}) = \underset{i=1}{\overset{n}{\pi}} f(x) = \underset{i=1}{\overset{n}{\pi}} \frac{e^{-\theta}\theta^x}{x!} = \frac{e^{-n\theta}\theta^{\sum x}}{\underset{i=1}{\overset{n}{\pi}} x!}$$

Let by definition pitman estimator for scale

$$T(X_1, X_2, X_3, \ldots, X_n) = \frac{\int_0^{\infty} \frac{1}{\theta^2} \frac{e^{-n\theta}\theta^{\sum x}}{\underset{i=1}{\overset{n}{\pi}} x!} d\theta}{\int_0^{\infty} \frac{1}{\theta^3} \frac{e^{-n\theta}\theta^{\sum x}}{\underset{i=1}{\overset{n}{\pi}} x!} d\theta}$$

$$T(X_1, X_2, X_3, \ldots, X_n) = \frac{\int_0^{\infty} \frac{1}{\theta^2} e^{-n\theta}\theta^{\sum x} d\theta}{\int_0^{\infty} \frac{1}{\theta^3} e^{-n\theta}\theta^{\sum x} d\theta}$$

$$T(X_1, X_2, X_3, \ldots, X_n) = \frac{\int_0^{\infty} e^{-n\theta}\theta^{\sum x-2} d\theta}{\int_0^{\infty} e^{-n\theta}\theta^{\sum x-3} d\theta}$$

$$T(X_1, X_2, X_3, \ldots, X_n) = \frac{\int_0^{\infty} \theta^{\sum x-2} e^{-\theta/n^{-1}} d\theta}{\int_0^{\infty} \theta^{\sum x-3} e^{-\theta/n^{-1}} d\theta} \qquad (A)$$

As we know that Gamma function is

$$\sqrt{\alpha}\,\beta^{\alpha} = \int_{0}^{\infty} \theta^{\alpha-1} e^{-\theta/\beta} d(\theta) \qquad \text{(B)}$$

Comparing (A) and (B)

$$T(X_1, X_2, X_3, ..., X_n) = \frac{\sqrt{\sum x-2}\,(n^{-1})^{\sum x-2}}{\sqrt{\sum x-3}\,(n^{-1})^{\sum x-3}}$$

$$T(X_1, X_2, X_3, ..., X_n) = \frac{\sqrt{\sum x-3+1}\,(n^{-1})^{\sum x-2}}{\sqrt{\sum x-2}\,(n^{-1})^{\sum x-3}}$$

$$T(X_1, X_2, X_3, ..., X_n) = \frac{(\sum x-3)\sqrt{\sum x-3}}{\sqrt{\sum x-3}\,n}$$

$$T(X_1, X_2, X_3, ..., X_n) = \frac{(\sum x-3)}{n}$$

$$T(X_1, X_2, X_3, ..., X_n) = \frac{(n\bar{X}-3)}{n}$$